Derivation of Local Volatility

by Fabrice Douglas Rouah
www.FRouah.com
www.Volopta.com

The derivation of local volatility is outlined in many papers and textbooks (such as the one by Jim Gatheral [1]), but in the derivations many steps are left out. In this Note we provide two derivations of local volatility.

1. The derivation by Dupire [2] that uses the Fokker-Planck equation.
2. The derivation by Derman *et al.* [3] of local volatility as a conditional expectation.

We also present the derivation of local volatility from Black-Scholes implied volatility, outlined in [1]. We will derive the following three equations that involve local volatility $\sigma = \sigma(S_t, t)$ or local variance $v_L = \sigma^2$.

1. The Dupire equation in its most general form (appears in [1] on page 9)

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + (r_T - q_T)\left(C - K\frac{\partial C}{\partial K}\right) - r_T C. \tag{1}$$

2. The equation by Derman *et al.* [3] for local volatility as a conditional expected value (appears with $q_T = 0$ in [3])

$$\frac{\partial C}{\partial T} = -K(r_T - q_T)\frac{\partial C}{\partial K} - q_T C + \frac{1}{2}K^2 E\left[\sigma_T^2 | S_T = K\right]\frac{\partial^2 C}{\partial K^2}. \tag{2}$$

3. Local volatility as a function of Black-Scholes implied volatility, $\Sigma = \Sigma(K, T)$ (appears in [1]) expressed here as the local variance v_L

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left[1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w}\right)\left(\frac{\partial w}{\partial y}\right)^2\right]}. \tag{3}$$

where $w = \Sigma(K, T)^2 T$ is the Black-Scholes total implied variance and $y = \ln\frac{K}{F_T}$ where $F_T = \exp\left(\int_0^T \mu_t dt\right)$ is the forward price with $\mu_t = r_t - q_t$ (risk free rate minus dividend yield). Alternatively, local volatility can also be expressed in terms of Σ as

$$\frac{\Sigma^2 + 2\Sigma T\left[\frac{\partial \Sigma}{\partial T} + (r_T - q_T)K\frac{\partial \Sigma}{\partial K}\right]}{\left(1 + \frac{Ky}{\Sigma}\frac{\partial \Sigma}{\partial K}\right)^2 + K\Sigma T\left[\frac{\partial \Sigma}{\partial K} - \frac{1}{4}K\Sigma T\left(\frac{\partial \Sigma}{\partial K}\right)^2 + K\frac{\partial^2 \Sigma}{\partial K^2}\right]}.$$

Solving for the local variance in Equation (1), we obtain

$$\sigma^2 = \sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T} - (r_T - q_T)\left(C - K\frac{\partial C}{\partial K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \tag{4}$$

If we set the risk-free rate r_T and the dividend yield q_T each equal to zero, Equations (1) and (2) can each be solved to yield the same equation involving local volatility, namely

$$\sigma^2 = \sigma (K,T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}. \tag{5}$$

The local volatility is then $v_L = \sqrt{\sigma^2(K,T)}$. In this Note the derivation of these equations are all explained in detail.

1 Local Volatility Model for the Underlying

The underlying S_t follows the process

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma(S_t,t) S_t dW_t \\ &= (r_t - q_t) S_t dt + \sigma(S_t,t) S_t dW_t. \end{aligned} \tag{6}$$

We sometimes drop the subscript and write $dS = \mu S dt + \sigma S dW$ where $\sigma = \sigma(S_t,t)$. We need the following preliminaries:

- Discount factor $P(t,T) = \exp\left(-\int_t^T r_s ds\right)$.
- Fokker-Planck equation. Denote by $f(S_t,t)$ the probability density function of the underlying price S_t at time t. Then f satisfies the equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial S}\left[\mu S f(S,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,t)\right]. \tag{7}$$

- Time-t price of European call with strike K, denoted $C = C(S_t,K)$

$$\begin{aligned} C &= P(t,T) E\left[(S_T - K)^+\right] \\ &= P(t,T) E\left[(S_T - K)\mathbf{1}_{(S_T>K)}\right] \\ &= P(t,T)\int_K^\infty (S_T - K) f(S,T) dS. \end{aligned} \tag{8}$$

where $\mathbf{1}_{(S_T>K)}$ is the Heaviside function and where $E[\cdot] = E[\cdot|\mathcal{F}_t]$. In the all the integrals in this Note, since the expectations are taken for the underlying price at $t = T$ it is understood that $S = S_T$, $f(S,T) = f(S_T,T)$ and $dS = dS_T$. We sometimes omit the subscript for notational convenience.

2 Derivation of the General Dupire Equation (1)

2.1 Required Derivatives

We need the following derivatives of the call $C(S_t,t)$.

- First derivative with respect to strike

$$\begin{aligned}
\frac{\partial C}{\partial K} &= P(t,T)\int_K^\infty \frac{\partial}{\partial K}(S_T-K)f(S,T)dS \qquad (9)\\
&= -P(t,T)\int_K^\infty f(S,T)dS.
\end{aligned}$$

- Second derivative with respect to strike

$$\begin{aligned}
\frac{\partial^2 C}{\partial K^2} &= -P(t,T)\left[f(S,T)\right]_{S=K}^{S=\infty} \qquad (10)\\
&= P(t,T)f(K,T).
\end{aligned}$$

We have assumed that $\lim_{S\to\infty} f(S,T)=0$.

- First derivative with respect to maturity–use the chain rule

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{\partial C}{\partial T}P(t,T)\times\int_K^\infty (S_T-K)f(S,T)dS + \qquad (11)\\
& P(t,T)\times\int_K^\infty (S_T-K)\frac{\partial}{\partial T}\left[f(S,T)\right]dS.
\end{aligned}$$

Note that $\frac{\partial P}{\partial T}=-r_T P(t,T)$ so we can write (11)

$$\frac{\partial C}{\partial T} = -r_T C + P(t,T)\int_K^\infty (S_T-K)\frac{\partial}{\partial T}\left[f(S,T)\right]dS. \qquad (12)$$

2.2 Main Equation

In Equation (12) substitute the Fokker-Planck equation (7) for $\frac{\partial f}{\partial t}$ at $t=T$

$$\begin{aligned}
\frac{\partial C}{\partial T}+r_T C &= P(t,T)\int_K^\infty (S_T-K)\times \qquad (13)\\
&\left\{-\frac{\partial}{\partial S}\left[\mu_T S f(S,T)\right]+\frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,T)\right]\right\}dS.
\end{aligned}$$

This is the main equation we need because it is from this equation that the Dupire local volatility is derived. In Equation (13) have two integrals to evaluate

$$\begin{aligned}
I_1 &= \mu_T\int_K^\infty (S_T-K)\frac{\partial}{\partial S}\left[Sf(S,T)\right]dS, \qquad (14)\\
I_2 &= \int_K^\infty (S_T-K)\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,T)\right]dS.
\end{aligned}$$

Before evaluating these two integrals we need the following two identities.

2.3 Two Useful Identities

2.3.1 First Identity

From the call price Equation (8), we obtain

$$\begin{aligned}\frac{C}{P(t,T)} &= \int_K^\infty (S_T - K) f(S,T) dS \qquad (15)\\ &= \int_K^\infty S_T f(S,T) dS - K \int_K^\infty f(S,T) dS.\end{aligned}$$

From the expression for $\frac{\partial C}{\partial K}$ in Equation (9) we obtain

$$\int_K^\infty f(S,T) dS = -\frac{1}{P(t,T)} \frac{\partial C}{\partial K}.$$

Substitute back into Equation (15) and re-arrange terms to obtain the first identity

$$\int_K^\infty S_T f(S,T) dS = \frac{C}{P(t,T)} - \frac{K}{P(t,T)} \frac{\partial C}{\partial K}. \qquad (16)$$

2.3.2 Second Identity

We use the expression for $\frac{\partial^2 C}{\partial K^2}$ in Equation (10) to obtain the second identity

$$f(K,T) = \frac{1}{P(t,T)} \frac{\partial^2 C}{\partial K^2}. \qquad (17)$$

2.4 Evaluating the Integrals

We can now evaluate the integrals I_1 and I_2 defined in Equation (14).

2.4.1 First integral

Use integration by parts with $u = S_T - K, u' = 1, v' = \frac{\partial}{\partial S}\left[S f(S,T)\right], v = S f(S,T)$

$$\begin{aligned}I_1 &= \left[\mu_T (S_T - K) S_T f(S,T)\right]_{S=K}^{S=\infty} - \mu_T \int_K^\infty S f(S,T) dS\\ &= [0 - 0] - \mu_T \int_K^\infty S f(S,T) dS.\end{aligned}$$

We have assumed $\lim_{S\to\infty} (S - K) S f(S,T) = 0$. Substitute the first identity (16) to obtain the first integral I_1

$$I_1 = \frac{-\mu_T C}{P(t,T)} + \frac{\mu_T K}{P(t,T)} \frac{\partial C}{\partial K}. \qquad (18)$$

2.4.2 Second integral

Use integration by parts with $u = S_T - K, u' = 1, v' = \frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,T)\right], v = \frac{\partial}{\partial S}\left[\sigma^2 S^2 f(S,T)\right]$

$$\begin{aligned} I_2 &= \left[(S_T - K)\frac{\partial}{\partial S}\left\{\sigma^2 S^2 f(S,T)\right\}\right]_{S=K}^{S=\infty} - \int_K^\infty \frac{\partial}{\partial S}\left[\sigma^2 S^2 f(S,T)\right] dS \\ &= [0-0] - \left[\sigma^2 S^2 f(S,T)\right]_{S=K}^{S=\infty} \\ &= \sigma^2 K^2 f(K,T) \end{aligned}$$

where $\sigma^2 = \sigma(K,T)^2$. We have assumed that $\lim_{S\to\infty} \frac{\partial}{\partial S}\left\{\sigma^2 S^2 f(S,T)\right\} = 0$. Substitute the second identity (17) for $f(K,T)$ to obtain the second integral I_2

$$I_2 = \frac{\sigma^2 K^2}{P(t,T)} \frac{\partial^2 C}{\partial K^2}. \tag{19}$$

2.5 Obtaining the Dupire Equation

We can now evaluate the main Equation (13) which we write as

$$\frac{\partial C}{\partial T} + r_T C = P(t,T)\left[-I_1 + \frac{1}{2}I_2\right].$$

Substitute for I_1 from Equation (18) and for I_2 from Equation (19)

$$\frac{\partial C}{\partial T} + r_T C = \mu_T C - \mu_T K \frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

Substitute for $\mu_T = r_T - q_T$ (risk free rate minus dividend yield) to obtain the Dupire equation (1)

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + (r_T - q_T)\left(C - K\frac{\partial C}{\partial K}\right) - r_T C.$$

Solve for $\sigma^2 = \sigma(K,T)^2$ to obtain the Dupire local variance in its general form

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T} + q_T C + (r_T - q_K) K \frac{\partial C}{\partial K}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}$$

Dupire [2] assumes zero interest rates and zero dividend yield. Hence $r_T = q_T = 0$ so that the underlying process is $dS_t = \sigma(S_t,t)S_t dW_t$. We obtain

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}.$$

which is Equation (5).

3 Derivation of Local Volatility as an Expected Value, Equation (2)

We need the following preliminaries, all of which are easy to show

$\frac{\partial}{\partial S}(S-K)^+ = \mathbf{1}_{(S>K)}$	$\frac{\partial}{\partial S}\mathbf{1}_{(S>K)} = \delta(S-K)$
$\frac{\partial}{\partial K}(S-K)^+ = -\mathbf{1}_{(S>K)}$	$\frac{\partial}{\partial K}\mathbf{1}_{(S>K)} = -\delta(S-K)$
$\frac{\partial C}{\partial K} = -P(t,T)E\left[\mathbf{1}_{(S>K)}\right]$	$\frac{\partial^2 C}{\partial K^2} = P(t,T)E\left[\delta(S-K)\right]$

In the table, $\delta(\cdot)$ denotes the Dirac delta function. Now define the function $f(S_T, T)$ as

$$f(S_T,T) = P(t,T)(S_T-K)^+.$$

Recall the process for S_t is given by Equation (6). By Itō's Lemma, f follows the process

$$df = \left[\frac{\partial f}{\partial T} + \mu_T S_T \frac{\partial f}{\partial S_T} + \frac{1}{2}\sigma_T^2 S_T \frac{\partial^2 f}{\partial S_T^2}\right] dT + \left[\sigma_T S_T \frac{\partial f}{\partial S_T}\right] dW_T. \tag{20}$$

Now the partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial T} &= -r_T P(t,T)(S_T-K)^+, \\
\frac{\partial f}{\partial S_T} &= P(t,T)\mathbf{1}_{(S_T>K)}, \\
\frac{\partial^2 f}{\partial S_T^2} &= P(t,T)\delta\left(S_T-K\right).
\end{aligned}$$

Substitute them into Equation (20)

$$\begin{aligned}
df &= P(t,T)\times \qquad\qquad (21)\\
&\quad \left[-r_T(S_T-K)^+ + \mu_T S_T \mathbf{1}_{(S_T>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S_T-K)\right] dT \\
&\quad +P(t,T)\left[\sigma_T S_T \mathbf{1}_{(S_T>K)}\right] dW_T
\end{aligned}$$

Consider the first two terms of (21), which can be written as

$$\begin{aligned}
-r_T(S_T-K)^+ + \mu_T S_T \mathbf{1}_{(S_T>K)} &= -r_T(S_T-K)\mathbf{1}_{(S_T>K)} + \mu_T S_T \mathbf{1}_{(S_T>K)} \\
&= r_T K \mathbf{1}_{(S_T>K)} - q_T S_T \mathbf{1}_{(S_T>K)}.
\end{aligned}$$

When we take the expected value of Equation (21), the stochastic term drops out since $E\left[dW_T\right] = 0$. Hence we can write the expected value of (21) as

$$\begin{aligned}
dC &= E\left[df\right] \qquad\qquad (22)\\
&= P(t,T)E\left[r_T K \mathbf{1}_{(S_T>K)} - q_T S_T \mathbf{1}_{(S_T>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S_T-K)\right] dT
\end{aligned}$$

so that

$$\frac{dC}{dT} = P(t,T)E\left[r_T K\mathbf{1}_{(S_T>K)} - q_T S_T\mathbf{1}_{(S_T>K)} + \frac{1}{2}\sigma_T^2 S_T^2\delta(S_T-K)\right]. \quad (23)$$

Using the second line in Equation (8), we can write

$$P(t,T)E\left[S_T\mathbf{1}_{(S_T>K)}\right] = C + KP(t,T)E\left[\mathbf{1}_{(S_T>K)}\right]$$

so Equation (23) becomes

$$\begin{aligned}
\frac{dC}{dT} &= KP(t,T)r_T E[\mathbf{1}_{(S_T>K)}] - q_T\left(C + KP(t,T)E\left[\mathbf{1}_{(S_T>K)}\right]\right) \quad (24)\\
&\quad + \frac{1}{2}P(t,T)E\left[\sigma_T^2 S_T^2\delta(S_T-K)\right]\\
&= -K(r_T-q_T)\frac{\partial C}{\partial K} - q_T C + \frac{1}{2}P(t,T)E\left[\sigma_T^2 S_T^2\delta(S_T-K)\right]
\end{aligned}$$

where we have substituted $-\frac{\partial C}{\partial K}$ for $P(t,T)E[\mathbf{1}_{(S_T>K)}]$. The last term in the last line of Equation (24) can be written

$$\begin{aligned}
\frac{1}{2}P(t,T)E\left[\sigma_T^2 S_T^2\delta(S_T-K)\right] &= \frac{1}{2}P(t,T)E\left[\sigma_T^2 S_T^2|S_T=K\right]E[\delta(S_T-K)]\\
&= \frac{1}{2}P(t,T)E\left[\sigma_T^2|S_T=K\right]K^2 E[\delta(S_T-K)]\\
&= \frac{1}{2}E\left[\sigma_T^2|S_T=K\right]K^2\frac{\partial^2 C}{\partial K^2}
\end{aligned}$$

where we have substituted $\frac{\partial^2 C}{\partial K^2}$ for $P(t,T)E[\delta(S_T-K)]$. We obtain the final result, Equation (2)

$$\frac{\partial C}{\partial T} = -K(r_T-q_T)\frac{\partial C}{\partial K} - q_T C + \frac{1}{2}K^2 E\left[\sigma_T^2|S_T=K\right]\frac{\partial^2 C}{\partial K^2}.$$

When $r_T = q_T = 0$ we can re-arrange the result to obtain

$$E\left[\sigma_T^2|S_T=K\right] = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

which, again, is Equation (5). Hence when the dividend and interest rate are both zero, the derivation of local volatility using Dupire's approach and the derivation using conditional expectation produce the same result.

4 Derivation of Local Volatility From Implied Volatility, Equation (3)

To express local volatility in terms of implied volatility, we need the three derivatives $\frac{\partial C}{\partial T}, \frac{\partial C}{\partial K}$, and $\frac{\partial^2 C}{\partial K^2}$ that appear in Equation (1), but expressed in terms of implied volatility. Following Gatheral [1] we define the log-moneyness

$$y = \ln \frac{K}{F_T}$$

where $F_T = S_0 \exp\left(\int_0^T \mu_t dt\right)$ is the forward price ($\mu_t = r_t - q_t$, risk free rate minus dividend yield) and K is the strike price, and the "total" Black-Scholes implied variance

$$w = \Sigma(K,T)^2 T$$

where $\Sigma(K,T)$ is the implied volatility. The Black-Scholes call price can then be written as

$$\begin{aligned} C_{BS}\left(S_0, K, \Sigma\left(K,T\right), T\right) &= C_{BS}\left(S_0, F_T e^y, w, T\right) \\ &= F_T\left\{N(d_1) - e^y N(d_2)\right\} \end{aligned} \tag{25}$$

where

$$d_1 = \frac{\ln\frac{S_0}{K} + \int_0^T \left(r_t - q_t\right) dt + \frac{w}{2}}{\sqrt{w}} = -y w^{-\frac{1}{2}} + \frac{1}{2} w^{\frac{1}{2}} \tag{26}$$

and $d_2 = d_1 - \sqrt{w} = -y w^{-\frac{1}{2}} - \frac{1}{2} w^{\frac{1}{2}}$.

4.1 The Reparameterized Local Volatility Function

To express the local volatility Equation (1) in terms of y, we note that the market call price is

$$C(S_0, K, T) = C(S_0, F_T e^y, T)$$

and we take derivatives. The first derivative we need is, by the chain rule

$$\frac{\partial C}{\partial y} = \frac{\partial C}{\partial K}\frac{\partial K}{\partial y} = \frac{\partial C}{\partial K} K. \tag{27}$$

The second derivative we need is

$$\begin{aligned} \frac{\partial^2 C}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial K}\right) K + \frac{\partial C}{\partial K}\frac{\partial K}{\partial y} \\ &= \frac{\partial^2 C}{\partial K^2} K^2 + \frac{\partial C}{\partial y}, \end{aligned} \tag{28}$$

since by the chain rule $\frac{\partial A}{\partial y} = \frac{\partial A}{\partial K}\frac{\partial K}{\partial y}$, so that $\frac{\partial}{\partial y}\left(\frac{\partial C}{\partial K}\right) = \frac{\partial^2 C}{\partial K^2}\frac{\partial K}{\partial y} = \frac{\partial^2 C}{\partial K^2} K$. The third derivative we need is

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K}\frac{\partial K}{\partial T} \\ &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K} K \mu_T \\ &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial y} \mu_T \end{aligned} \tag{29}$$

since $K = S_0 \exp\left(\int_0^T \mu_t dt + y\right)$ so that $\frac{\partial K}{\partial T} = K\mu_T$. Equation (28) implies that

$$\frac{\partial^2 C}{\partial K^2}K^2 = \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}.$$

Now we substitute into Equation (1), reproduced here for convenience

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + \mu_T\left(C - K\frac{\partial C}{\partial K}\right) \\
\frac{\partial C}{\partial T} - \frac{\partial C}{\partial y}\mu_T &= \frac{1}{2}\sigma^2\left(\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right) + \mu_T\left(C - \frac{\partial C}{\partial y}\right)
\end{aligned}$$

which simplifies to

$$\frac{\partial C}{\partial T} = \frac{v_L}{2}\left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right] + \mu_T C \tag{30}$$

where $v_L = \sigma^2(K,T)$ is the local variance. This is Equation (1.8) of Gatheral [1].

4.2 Three Useful Identities

Before expression the local volatility Equation (1) in terms of implied volatility, we first derive three identities used by Gatheral [1] that help in this regard. We use the fact that the derivatives of the standard normal cdf and pdf are, using the chain rule, $N'(x) = n(x)x'$ and $n'(x) = -xn(x)x'$. We also use the relation

$$\begin{aligned}
n(d_1) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_2+\sqrt{w}\right)^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_2^2+2d_2\sqrt{w}+w\right)} \\
&= n(d_2)e^{-d_2\sqrt{w}-\frac{1}{2}w} \\
&= n(d_2)e^{y}.
\end{aligned}$$

From Equation (25) the first derivative with respect to w is

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial w} &= F_T\left[n(d_1)d_{1w} - e^y n(d_2)d_{2w}\right] \\
&= F_T\left[n(d_2)e^y\left(d_{2w} + \frac{1}{2}w^{-\frac{1}{2}}\right) - e^y n(d_2)d_{2w}\right] \\
&= \frac{1}{2}F_T e^y\left[n(d_2)w^{-\frac{1}{2}}\right]
\end{aligned}$$

where d_{1w} is the first derivative of d_1 with respect to w and similarly for d_2. The second derivative is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial w^2} &= \frac{1}{2}F_T e^y \left[-n(d_2)d_2 d_{2w} w^{-\frac{1}{2}} - \frac{1}{2}n(d_2)w^{-\frac{3}{2}}\right] \\
&= \frac{1}{2}F_T e^y n(d_2) w^{-\frac{1}{2}}\left[-d_2 d_{2w} - \frac{1}{2}w^{-1}\right] \\
&= \frac{\partial C_{BS}}{\partial w}\left[\left(yw^{-\frac{1}{2}} + \frac{1}{2}w^{\frac{1}{2}}\right)\left(\frac{1}{2}yw^{-\frac{3}{2}} - \frac{1}{4}w^{-\frac{1}{2}}\right) - \frac{1}{2}w^{-1}\right] \\
&= \frac{\partial C_{BS}}{\partial w}\left[-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right].
\end{aligned} \tag{31}$$

This is the first identity we need. The second identity we need is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial w \partial y} &= \frac{1}{2}F_T w^{-\frac{1}{2}}\frac{\partial}{\partial y}\left[e^y n(d_2)\right] \\
&= \frac{1}{2}F_T w^{-\frac{1}{2}}\left[e^y n(d_2) - e^y n(d_2) d_2 d_{2y}\right] \\
&= \frac{\partial C_{BS}}{\partial w}\left[1 - d_2 d_{2y}\right] \\
&= \frac{\partial C_{BS}}{\partial w}\left(\frac{1}{2} - \frac{y}{w}\right)
\end{aligned} \tag{32}$$

where $d_{2y} = -w^{-\frac{1}{2}}$ is the first derivative of d_2 with respect to y. To obtain the third identity, consider the derivative

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial y} &= F_T\left[n(d_1)d_{1y} - e^y N(d_2) - e^y n(d_2) d_{2y}\right] \\
&= F_T e^y\left[n(d_2)d_{1y} - N(d_2) - n(d_2)d_{2y}\right] \\
&= -F_T e^y N(d_2).
\end{aligned}$$

The third identity we need is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial y^2} &= -F_T\left[e^y N(d_2) + e^y n(d_2) d_{2y}\right] \\
&= -F_T e^y N(d_2) + F_T e^y n(d_2) w^{-\frac{1}{2}} \\
&= \frac{\partial C_{BS}}{\partial y} + 2\frac{\partial C_{BS}}{\partial w}.
\end{aligned} \tag{33}$$

We are now ready for the main derivation of this section.

4.3 Local Volatility in Terms of Implied Volatility

We note that when the market price $C(S_0, K, T)$ is equal to the Black-Scholes price with the implied volatility $\Sigma(K, T)$ as the input to volatility

$$C(S_0, K, T) = C_{BS}(S_0, K, \Sigma(K, T), T). \tag{34}$$

We can also reparameterize the Black-Scholes price in terms of the total implied volatility $w = \Sigma(K,T)^2 T$ and $K = F_T e^y$. Since w depends on K and K depends on y, we have that $w = w(y)$ and we can write

$$C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T). \tag{35}$$

We need derivatives of the market call price $C(S_0, K, T)$ in terms of the Black-Scholes call price $C_{BS}(S_0, F_T e^y, w(y), T)$. From Equation (35), the first derivative we need is

$$\begin{aligned}\frac{\partial C}{\partial y} &= \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial y} \\ &= a(w,y) + b(w,y)c(y).\end{aligned} \tag{36}$$

It is easier to visualize the second derivative we need, $\frac{\partial^2 C}{\partial y^2}$, when we express the partials in $\frac{\partial C}{\partial y}$ as a, b, and c.

$$\begin{aligned}\frac{\partial^2 C}{\partial y^2} &= \frac{\partial a}{\partial y} + \frac{\partial a}{\partial w}\frac{\partial w}{\partial y} + b(w,y)\frac{\partial c}{\partial y} + \left[\frac{\partial b}{\partial y} + \frac{\partial b}{\partial w}\frac{\partial w}{\partial y}\right] c(y) \\ &= \frac{\partial^2 C_{BS}}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2} + \left[\frac{\partial^2 C_{BS}}{\partial w \partial y} + \frac{\partial^2 C_{BS}}{\partial w^2}\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial y} \\ &= \frac{\partial^2 C_{BS}}{\partial y^2} + 2\frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2.\end{aligned} \tag{37}$$

The third derivative we need is

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} \\ &= \mu_T C + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T}.\end{aligned} \tag{38}$$

Gatheral explains that the second equality follows because the only explicit dependence of C_{BS} on T is through the forward price F_T, even though C_{BS} depends implicitly on T through y and w. The reparameterized Dupire equation (30) is reproduced here for convenience

$$\frac{\partial C}{\partial T} = \frac{v_L}{2}\left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right] + \mu_T C.$$

We substitute for $\frac{\partial C}{\partial T}, \frac{\partial^2 C}{\partial y^2}$, and $\frac{\partial C}{\partial y}$ from Equations (38), (37), and (36) respectively and cancel $\mu_T C$ from both sides to obtain

$$\begin{aligned}\frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} &= \frac{v_L}{2}\left[\frac{\partial^2 C_{BS}}{\partial y^2} + 2\frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2\right. \\ &\qquad \left. - \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial y}\right].\end{aligned} \tag{39}$$

Now substitute for $\frac{\partial^2 C_{BS}}{\partial w^2}$, $\frac{\partial^2 C_{BS}}{\partial w \partial y}$, and $\frac{\partial^2 C_{BS}}{\partial y^2}$ from the identities in Equations (31), (32), and (33) respectively, the idea being to end up with terms involving $\frac{\partial C_{BS}}{\partial w}$ on the right hand side of Equation (39) that can be factored out.

$$\begin{aligned} \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} &= \frac{v_L}{2}\frac{\partial C_{BS}}{\partial w}\left[2+2\left(\frac{1}{2}-\frac{y}{w}\right)\frac{\partial w}{\partial y}+\left(-\frac{1}{8}-\frac{1}{2w}+\frac{y^2}{2w}\right)\left(\frac{\partial w}{\partial y}\right)^2 \right. \\ &\quad \left. +\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial y}\right]. \end{aligned}$$

Remove the factor $\frac{\partial C_{BS}}{\partial w}$ from both sides and simplify to obtain

$$\frac{\partial w}{\partial T} = v_L\left[1-\frac{y}{w}\frac{\partial w}{\partial y}+\frac{1}{2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^2}{w}\right)\left(\frac{\partial w}{\partial y}\right)^2\right].$$

Solve for v_L to obtain the final expression for the local volatility expressed in terms of implied volatility $w = \Sigma(K,T)^2 T$ and the log-moneyness $y = \ln \frac{K}{F_T}$

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left[1-\frac{y}{w}\frac{\partial w}{\partial y}+\frac{1}{2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^2}{w}\right)\left(\frac{\partial w}{\partial y}\right)^2\right]}.$$

4.4 Alternate Derivation

In this derivation we express the derivatives $\frac{\partial C}{\partial K}$, $\frac{\partial^2 C}{\partial K^2}$, and $\frac{\partial C}{\partial T}$ in the Dupire equation (1) in terms of y and $w = w(y)$, but we substitute these derivatives directly in Equation (1) rather than in (30). This means that we take derivatives with respect to K and T, rather than with y and T. Recall that from Equation (35), the market call price is equal to the Black-Scholes call price with implied volatility as input

$$C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T).$$

Recall also that from Equation (25) the Black-Scholes call price reparameterized in terms of y and w is

$$C_{BS}(S_0, F_T e^y, w(y), T) = F_T\left\{N(d_1) - e^y N(d_2)\right\}$$

where d_1 is given in Equation (26), and where $d_2 = d_1 - \sqrt{w}$. The first derivative

we need is

$$\begin{aligned} \frac{\partial C}{\partial K} &= \frac{\partial C_{BS}}{\partial y}\frac{\partial y}{\partial K}+\frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial K} \qquad (40) \\ &= \frac{1}{K}\frac{\partial C_{BS}}{\partial y}+\frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial K}. \end{aligned}$$

The second derivative is

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2} &= -\frac{1}{K^2}\frac{\partial C_{BS}}{\partial y} + \frac{1}{K}\frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial y}\right). \\ &\quad + \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial w}\right)\frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial K^2}\end{aligned} \tag{41}$$

Let $A = \frac{\partial C}{\partial y}$ for notational convenience. Then $\frac{\partial}{\partial K}\left(\frac{\partial C}{\partial y}\right) = \frac{\partial A}{\partial K}$ and

$$\begin{aligned}\frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial y}\right) &= \frac{\partial A}{\partial K} \\ &= \frac{\partial A}{\partial y}\frac{\partial y}{\partial K} + \frac{\partial A}{\partial w}\frac{\partial w}{\partial K} \\ &= \frac{\partial^2 C_{BS}}{\partial y^2}\frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial K}.\end{aligned} \tag{42}$$

Similarly

$$\frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial w}\right) = \frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2}\frac{\partial w}{\partial K}. \tag{43}$$

Substituting Equations (42) and (43) into Equation (41) produces

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2} &= -\frac{1}{K^2}\frac{\partial C_{BS}}{\partial y} + \frac{1}{K}\left(\frac{\partial^2 C_{BS}}{\partial y^2}\frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial K}\right) \\ &\quad + \left(\frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2}\frac{\partial w}{\partial K}\right)\frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial K^2} \\ &= \frac{1}{K^2}\left(\frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y}\right) + \frac{2}{K}\frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial K} \\ &\quad + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial K}\right)^2 + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial K^2}.\end{aligned} \tag{44}$$

The third derivative we need is

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial y}\frac{\partial y}{\partial T} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} \\ &= \mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y}\mu_T + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T},\end{aligned} \tag{45}$$

again using the fact that $\frac{\partial C_{BS}}{\partial T}$ depends explicitly on T only through F_T. Now substitute for $\frac{\partial C}{\partial K}$, $\frac{\partial^2 C}{\partial K^2}$, and $\frac{\partial C}{\partial T}$ from Equations (40), (44), and (45) respectively into Equation (4) for Dupire local variance, reproduced here for convenience.

$$\sigma^2 = \frac{\frac{\partial C}{\partial T} - \mu_T\left[C_{BS} - K\frac{\partial C}{\partial K}\right]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}.$$

We obtain, after applying the three useful identities in Section 4.2,

$$\sigma^2 = \frac{\mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y}\mu_T + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} - \mu_T\left[C_{BS} - K\left(\frac{1}{K}\frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial K}\right)\right]}{\frac{1}{2}K^2\left[\frac{1}{K^2}\left(\frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y}\right) + \frac{2}{K}\frac{\partial^2 C_{BS}}{\partial y \partial w}\frac{\partial w}{\partial K} + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial K}\right)^2 + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial K^2}\right]}.$$

Applying the three useful identities in Section 4.2 allows the term $\frac{\partial C_{BS}}{\partial w}$ to be factored out of the numerator and denominator. The last equation becomes

$$\sigma^2 = \frac{\left[\frac{\partial w}{\partial T} + \mu_T K \frac{\partial w}{\partial K}\right]}{\frac{1}{2}K^2\left[\frac{2}{K^2} + \frac{2}{K}\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial w}{\partial K} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\left(\frac{\partial w}{\partial K}\right)^2 + \frac{\partial^2 w}{\partial K^2}\right]}. \tag{46}$$

Equation (46) can be simplified by considering deriving the partial derivatives of the Black-Scholes total implied variance, $w = \Sigma(K,T)^2 T$. We have $\frac{\partial w}{\partial T} = 2\Sigma T \frac{\partial \Sigma}{\partial T} + \Sigma^2$, $\frac{\partial w}{\partial K} = 2\Sigma T \frac{\partial \Sigma}{\partial K}$, and $\frac{\partial^2 w}{\partial K^2} = 2T\left[\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \Sigma\frac{\partial^2 \Sigma}{\partial K^2}\right]$. Substitute into Equation (46). The numerator in Equation (46) becomes

$$\Sigma^2 + 2\Sigma T\left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K}\right) \tag{47}$$

and the denominator becomes

$$\begin{aligned}
&1 + 2K\Sigma T\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial \Sigma}{\partial K} + 2K^2\Sigma^2 T^2\left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\left(\frac{\partial \Sigma}{\partial K}\right)^2 \\
&+K^2 T\left[\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \Sigma\frac{\partial^2 \Sigma}{\partial K^2}\right].
\end{aligned}$$

Replacing w with $\Sigma^2 T$ everywhere in the denominator produces

$$\begin{aligned}
&1 + 2K\Sigma T\left(\frac{1}{2} - \frac{y}{\Sigma^2 T}\right)\frac{\partial \Sigma}{\partial K} + 2K^2\Sigma^2 T^2\left(-\frac{1}{8} - \frac{1}{2\Sigma^2 T} + \frac{y^2}{2\Sigma^4 T^2}\right)\left(\frac{\partial \Sigma}{\partial K}\right)^2 \\
&+K^2 T\left[\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \Sigma\frac{\partial^2 \Sigma}{\partial K^2}\right] \\
= \;& 1 + K\Sigma T\frac{\partial \Sigma}{\partial K} - \frac{2Ky}{\Sigma}\frac{\partial \Sigma}{\partial K} - \frac{K^2\Sigma^2 T^2}{4}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{K^2 y^2}{\Sigma^2}\left(\frac{\partial \Sigma}{\partial K}\right)^2 \\
&+K^2\Sigma T\frac{\partial^2 \Sigma}{\partial K^2} \\
= \;& \left(1 - \frac{Ky}{\Sigma}\frac{\partial \Sigma}{\partial K}\right)^2 + \left[1 - 2\frac{Ky}{\Sigma}\frac{\partial \Sigma}{\partial K} + \left(\frac{Ky}{\Sigma}\frac{\partial \Sigma}{\partial K}\right)^2\right].
\end{aligned} \tag{48}$$

Substituting the numerator in (47) and the denominator in (48) back to Equation (46), we obtain

$$\frac{\Sigma^2 + 2\Sigma T\left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K}\right)}{\left(1 + \frac{Ky}{\Sigma}\frac{\partial \Sigma}{\partial K}\right)^2 + K\Sigma T\left[\frac{\partial \Sigma}{\partial K} - \frac{1}{4}K\Sigma T\left(\frac{\partial \Sigma}{\partial K}\right)^2 + K\frac{\partial^2 \Sigma}{\partial K^2}\right]}$$

See also the dissertation by van der Kamp [4] for additional details of this alternate derivation.

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